GENERAL SOLUTIONS FOR CHOICE SETS: THE GENERALIZED OPTIMAL-CHOICE AXIOM SET

ATHANASIOS ANDRIKOPOULOS

Abstract. A characterization of the Generalized Optimal-Choice Axiom (Schwartz) set of a binary relation in terms of choice from maximal consistent subrelations, over non-finite sets is presented. Schwartz's results in [2, Theorems 6.2.1 - 6.2.2], for asymmetric binary relations stated in the finite case, are extended to this more general framework.

JEL classification: D70.

Key words: Generalized Optimal-Choice Axiom, maximal elements, acyclicity, consistency.

1. Introduction

One of the most common models employed in economics and social sciences is that of describing individual or group choices by means of a maximization problem: the individual makes choices by selecting, from each feasible set of alternatives, those which maximize his own preference relation. In the literature on social preferences, there are a great number of papers devoted to the analysis of the existence of maximal elements, as they are considered the best ones. Generally, the existence of maximal elements cannot be ensured, but if there is a maximal element, it will be the one that is preferable to all other alternatives (Condorcet winner). When the set of maximal choices (maximal choice set) is empty[1], the crucial question which has been arisen, is what to count as a choice. That is, what sets of alternatives may be considered as reasonable solutions of choice problems when maximal choices are absent? To answer this question, several methods (solution theories) for constructing non-empty choice sets have been proposed. Such a solution is the concept of *Generalized Optimal-Choice Axiom set* ($\mathcal{GOCHA}$ *set*), which is a generalization of the maximal choice set. The origin of $\mathcal{GOCHA}$ set can be found in Schwartz (see [2, pp. 145-147]), where it was shown that if the set of alternatives is finite, then: (a) (See [2, Theorem 6.2.1]). The $\mathcal{GOCHA}$ set of an asymmetric binary relation R is equivalent to the union of top cycles and maximal elements of R. (b) (See [2, Theorem 6.2.2]). The $\mathcal{GOCHA}$ set of an asymmetric binary relation R is equivalent to the set of maximal elements of the transitive closure of R.

[1]This problem is common in the analysis of pairwise majority voting, in the choice of a winning sport team, in the aggregation of multiple choice criteria, in committee selection, in choice under uncertainty or in decision under risk, etc.

In this paper, I show that an alternative belongs to the $\mathcal{GOCHA}$ set of an arbitrary binary relation R if and only if it is maximal for a maximal consistent subrelation of the transitive closure of R. I also extend the above mentioned results of Schwartz ([2, Theorems 6.2.1 - 6.2.2]) to arbitrary binary relations defined in infinite sets of alternatives.

2. Notation and definitions

Let X be a non-empty universal set of alternatives, and let $R \subseteq X \times X$ be a binary relation on X. We sometimes abbreviate $(x,y) \in R$ as xRy. For any $x \in X$, $Rx = \{y \in X \mid yRx\}$ and $xR = \{y \in X \mid xRy\}$ denote respectively the *upper contour* and *lower contour* of R *at* x. The *asymmetric part* $P(R)$ of R is given by: $(x,y) \in P(R)$ if and only if $(x,y) \in R$ and $(y,x) \notin R$. $\mathcal{M}(R)$ denote the elements of X that are maximal with respect to R in X, i.e., $\mathcal{M}(R) = \{x \in X|$ for all $y \in X, yRx$ implies $xRy\}$. The *transitive closure* of R is denoted by $\overline{R}$, that is for all $x, y \in X, (x,y) \in \overline{R}$ if there exist $k \in \mathbb{N}$ and $x_0, ..., x_K \in X$ such that $x = x_0$, $(x_{k-1}, x_k) \in R$ for all $k \in \{1, ..., K\}$ and $x_K = y$. A subset $Y \subseteq X$ is a cycle if, for all $x, y \in Y$, we have $(x,y) \in \overline{R}$ and $(y,x) \in \overline{R}$. The subset Y is a *top cycle* if there is no $x \in X \setminus Y$ and there is no $y \in Y$ such that $(x,y) \in R$. A subset Y of X is *undominated* if there is no alternative outside Y that is preferable to an alternative inside Y. If the subset $Y = \{x\}$ is a singleton, then x is called an *undominated element* of X. An undominated set is *minimal* if none of its proper subsets has this property. The binary relation R is *consistent*,[2] if for all $x, y \in X$, for all $k \in \mathbb{N}$, and for all $x_0, x_1, ..., x_K \in X$, if $x = x_0$, $(x_{k-1}, x_k) \in R$ for all $k \in \{1, ..., K\}$ and $x_{_K} = y$, then $(y,x) \notin P(R)$.

Let Ω be a family of non-empty subsets of X that represents the different feasible sets presented for choice. A choice function is a mapping that assigns to each choice situation a subset of it:

$$C : \Omega \to X \text{ such that for all } A \in \Omega,\ C(A) \subseteq A.$$

Best choices can be expressed as the maximization of the individuals's preferences over a set of alternatives. That is, for every $A \in \Omega$, $C(A) = \mathcal{M}(R/A)$ ($\mathcal{M}(R/A)$ denote the elements of X that are maximal for R in A). To deal with case where the set of maximal choices $C(A)$ is empty, Schwartz [4, Definition in page 141] has been proposed the following general solution:

The Generalized Optimal-Choice Axiom $\mathcal{GOCHA}$: For each $A \in \Omega$, $C(A)$ is equivalent to the union of minimum undominated subsets of A. The $\mathcal{GOCHA}$ *set* is the choice set from a given set specified by the $\mathcal{GOCHA}$ condition.

3. Main Result

The main result in this paper establishes a binary characterization of the choices generated by consistent subrelations. Let $\mathcal{A}_c$ denote the consistent subrelations

[2]Consistency is a central property for the existence of an ordering extension, for the analysis of choice, for the existence of a social welfare ordering in the sense of Bergson and Samuelson (see [3]).

of $\overline{R}$, i.e., $\mathcal{A}_c = \{Q \subseteq \overline{R} \mid Q \text{ is consistent}\}$, and let $\mathcal{A}_c^{\mathcal{M}}$ denote the elements of $\mathcal{A}_c$ that are maximal with respect to set inclusion.

Proposition 1. Let X be a nonempty set of alternatives and let R be a binary relation over X. For each $x \in X$, there exists a consistent subrelation $R_{c(x)} \subseteq \overline{R}$ on $x\overline{R} \cup \{x\}$ such that $x\overline{R_{c(x)}} = x\overline{R} \setminus \{x\}$.

Proof. It is an immediate consequence of [1, Lemma 1]. □

The next result show the connection between the
$\mathcal{GOCHA}$ set of a binary relation R with the choice sets generated from consistent subrelations of $\overline{R}$. It is also obtains a generalization of the above mentioned results of Schwartz in [2, Theorems 6.2.1 - 6.2.2].

Theorem 2. Let X be a nonempty set of alternatives and let R be a binary relation over X. Then, the following sets are identical:
(a) the Generalized Optimal-Choice Axiom set,
(b) the union of all undominated elements of R and all top cycles in R,
(c) the union of maximal elements of $\overline{R}$,
(d) the set of maximal elements of all maximal consistent subrelations of $\overline{R}$.

Proof. $a \Rightarrow b$). Take any $x \in \mathcal{GOCHA}$ set. Then, there exists a minimal undominated subset D of X such that $x \in D$. Suppose that x is not undominated with respect to R. Then, there exists $y \in X$ such that $(y,x) \in R$. It follows that $y \in D$. Let $A_x = \{t \in D \mid (x,t) \in \overline{R}\ \}$. We have that $A_x \neq \emptyset$, because otherwise, for each $t \in D$, $(x,t) \notin \overline{R} \supseteq R$, which implies that $D \setminus \{x\} \subset D$ is an undominated subset of X, a contradiction because of the minimal character of D. Let $D^* = D \setminus A_x$. We now show that $D^* = \emptyset$. We proceed by the way of contradiction. Suppose that $D^* \neq \emptyset$. Then, for each $t \in A_x$ and each $s \in D^*$ we have $(t,s) \notin R$ for suppose otherwise, $(t,s) \in R$ implies that $(x,s) \in \overline{R}$ contradicting $s \in G^*$. Therefore, $D^* \subset D$ is an undominated subset of X, again a contradiction. Hence, $A_x = D$. Since, $y \in D$ we conclude that $(x,y) \in \overline{R}$ which jointly to $(y,x) \in R$ concludes that D is a cycle in X. Finally, since D is an undominated subset of X, we conclude that it is a top cycle in X.

$b \Rightarrow c$). We have two cases to consider: either x is an undominated element of X with respect to R, or else x belongs to a top cycle in X. In the first case, x also is an an undominated element of X with respect to $\overline{R}$. If instead, x belongs to a top cycle in X, then for each $y \in X$, either $(y,x) \notin \overline{R}$ or $(x,y) \in \overline{R}$ and $(y,x) \in \overline{R}$. The rest is obvious.

$c \Rightarrow d$). Take any x which belongs to $\mathcal{M}(\overline{R})$. We show that there exists $Q^* \in \mathcal{A}_c^{\mathcal{M}}$ such that $x \in \mathcal{M}(Q^*)$. Let $R_{c(x)}$ be as in Proposition 1. First, observe that x is a maximal element of $R_{c(x)}$ in $Y = x\overline{R} \cup \{x\}$. Indeed, suppose there exists $y \in Y$ such that $(y,x) \in P(R_{c(x)})$. Hence, $y \in Y \setminus \{x\} = x\overline{R_{c(x)}}$. This implies $(x,y) \in \overline{R_{c(x)}}$ contradicting consistency of $R_{c(x)}$. If $R_{c(x)}$ is maximal with respect to set-inclusion in X, then the proof is over. Otherwise, there exists at least one consistent subrelation Q such that $R_{c(x)} \subseteq Q \subseteq \overline{R}$. Let $\mathcal{Q}$

be the set of consistent subrelations Q satisfying the latter condition. Let $\mathcal{C}$ be a chain in $\mathcal{Q}$, and let $\mathcal{D}=\bigcup\mathcal{C}$. Evidently, $R_{c(x)}\subseteq\mathcal{D}\subseteq\overline{R}$. It follows that $\mathcal{D}$ is consistent, for suppose otherwise there exist a natural number m, $s\in X$ and alternatives $z_0,z_1,...,z_m\in X$ such that

$$s=z_0\mathcal{D}z_1...z_{m-1}\mathcal{D}z_mP(\mathcal{D})z_0=s.$$

Consider the largest Q for which there exist such $s,m,z_0,...,z_m$. Then, Q is non consistent, a contradiction. Therefore, by Zorn's lemma, $\mathcal{Q}$ has an element, say Q^*, that is maximal with respect to set inclusion. We prove that x is a maximal element for Q^*. We proceed by way of contradiction. Let $y\in X$ be such that $(y,x)\in P(Q^*)\subseteq\overline{R}$. Since $x\in\mathcal{M}(\overline{R})$, we must then have $(x,y)\in\overline{R}$. Thus, $y\in Y\setminus\{x\}=x\overline{R_{c(x)}}$. Therefore, we have $(x,y)\in\overline{R_{c(x)}}$, which implies $(x,y)\in\overline{Q^*}$, contradicting consistency of Q^*. Hence, $x\in\mathcal{M}(Q^*)$.

$d\Rightarrow a$). Let $Q^*\in\mathcal{A}_a^{\mathcal{M}}$ and let $x\in\mathcal{M}(Q^*)$. Then, for each $y\in X$ there holds $(y,x)\notin P(Q^*)$. We prove that $x\in\mathcal{GOCHA}$ set. If x is an undominated element of X, then the proof is over. Otherwise, there exists $y^*\in X$ such that $(y^*,x)\in R$. We have two cases to consider: (i) $(y^*,x)\notin Q^*$; (ii) $(x,y^*)\in Q^*$ and $(y^*,x)\in Q^*$. For the first case, let us define $Q_0^*=Q^*\cup\{(y^*,x)\}$. Clearly, $Q_0^*\subseteq\overline{R}$ and Q_0^* is non consistent (the assumption that Q_0^* is consistent contradicts to the fact that Q_0^* is maximal with respect to set-inclusion). Hence, there exist $s,t\in X$, $\lambda\in\mathbb{N}$, and $z_0,z_1,...,z_\Lambda\in X$ such that $s=z_0$, $(z_{\lambda-1},z_\lambda)\in Q_0^*$ for all $\lambda\in\{1,...,\Lambda\}$, $z_\Lambda=t$ and $(t,s)\in P(Q_0^*)\subseteq Q_0^*$. Since Q^* is consistent, there must exists $\lambda_0\in\{1,...,\Lambda\}$ such that $(z_{\lambda_0-1},z_{\lambda_0})=(y^*,x)$ and for all $\lambda\in\{1,...,\Lambda\}$ with $\lambda\neq\lambda_0$, $(z_{\lambda-1},z_\lambda)\in Q_0^*$ if and only if $(z_{\lambda-1},z_\lambda)\in Q_0^*$. It then follows that $(z_{\lambda_0},z_{\lambda_0-1})\in\overline{Q_0^*}$. Therefore, $(x,y^*)\in\overline{Q^*}\subseteq\overline{R}$ which jointly to $(y^*,x)\in R$ implies that x belongs to a Cycle in X. Let $\mathcal{C}(x)$ be the cycle containing x that is maximal in the sense that it is not a proper subset of any other cycle. We prove that $\mathcal{C}(x)$ is a minimal undominated subset of X. Suppose on the contrary, that $(t,z)\in R$ for some $t\in X\setminus\mathcal{C}(x)$ and $z\in\mathcal{C}(x)$; to deduce a contradiction. Hence, $(t,x)\in\overline{R}$. Since $x\in\mathcal{M}(Q^*)$, it follows that $(t,x)\notin P(Q^*)$. There are two subcases to consider: (i_α) $(x,t)\in Q^*\subseteq R$ and $(t,x)\in Q^*\subseteq R$; (i_β) $(t,x)\notin Q^*$. In subcase (i_α), it follows from $(t,x)\in\overline{R}$ that $\mathcal{C}(x)\cup\{t\}$ is a cycle, a contradiction. In subcase (i_β), as in the proof of the above case $(y^*,x)\notin Q^*$, we conclude that $(x,t)\in Q^*\subseteq\overline{R}$, again a contradiction, because of the maximal character of $\mathcal{C}(x)$. Clearly, by construction $\mathcal{C}(x)$ is a minimal undominated subset of X. It remains to prove the case (ii). But when $(x,y^*)\in Q^*$ and $(y^*,x)\in Q^*$, we have that x belongs to a cycle. The rest is similar to the case (i). The last conclusion completes the proof of the theorem. □

References

[1] Duggan, J. (2007). A systematic approach to the construction of non-empty Choice sets, *Social Choice and Welfare*, **28**, 491-506.

[2] Schwartz, T., The logic of Collective Choice, New York: Columbia University Press.

[3] Suzumura, K. (1976). Remarks on the theory of collective. *Economica*, **43**, 381-390.